

SMITH IDEALS OF OPERADIC ALGEBRAS IN MONOIDAL MODEL CATEGORIES

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ABSTRACT. Building upon Hovey's work on Smith ideals for monoids, we develop a homotopy theory of Smith ideals for general operads in a symmetric monoidal category. For a sufficiently nice stable monoidal model category and an operad satisfying a cofibrancy condition, we show that there is a Quillen equivalence between a model structure on Smith ideals and a model structure on algebra maps induced by the cokernel and the kernel. For symmetric spectra this applies to the commutative operad and all Sigma-cofibrant operads. For chain complexes over a field of characteristic zero and the stable module category, this Quillen equivalence holds for all operads.

1. INTRODUCTION

A major part of stable homotopy theory is the study of structured ring spectra. These include ring spectra, commutative ring spectra, A_∞ -ring spectra, E_∞ -ring spectra, and so forth. Based on an unpublished talk by Jeff Smith, in [Hov ∞] Hovey developed a homotopy theory of Smith ideals for ring spectra and monoids in more general symmetric monoidal model categories.

Let us briefly recall Hovey's work in [Hov ∞]. For a symmetric monoidal closed category M , its arrow category \vec{M} is the category whose objects are maps in M and whose maps are commutative squares in M . It has two symmetric monoidal closed structures, namely, the tensor product monoidal structure \vec{M}^\otimes and the pushout product monoidal structure \vec{M}^\square . A monoid in \vec{M}^\square is a Smith ideal, and a monoid in \vec{M}^\otimes is a monoid map. If M is a model category, then \vec{M}^\otimes has the injective model structure $\vec{M}_{\text{inj}}^\otimes$, where weak equivalences and cofibrations are defined entrywise, and the category of monoid maps inherits a model structure from $\vec{M}_{\text{inj}}^\otimes$. Likewise, \vec{M}^\square has the projective model structure $\vec{M}_{\text{proj}}^\square$, where weak equivalences and fibrations are defined entrywise, and the category of Smith ideals inherits a model structure from $\vec{M}_{\text{proj}}^\square$. Surprisingly, when M is pointed (resp., stable), the cokernel and the kernel form a Quillen adjunction (resp., Quillen equivalence) between \vec{M}^\square and \vec{M}^\otimes and also between Smith ideals and monoid maps.

Since monoids are algebras over the associative operad, a natural question is whether there is a satisfactory theory of Smith ideals for algebras over other operads. For the commutative operad, the first author showed in [Whi ∞] that commutative Smith ideals in symmetric spectra, equipped with either the positive flat (stable) or the positive (stable) model structure, inherit a model structure. The purpose of this paper is to generalize Hovey's work to Smith ideals for general operads in monoidal model categories. For an operad \mathcal{O} we define a Smith \mathcal{O} -ideal as an algebra over an associated operad $\vec{\mathcal{O}}^\square$ in the arrow category $\vec{\mathcal{M}}^\square$. We will prove a precise version of the following result in Theorem 4.4.1.

Theorem A. *Suppose \mathcal{M} is a sufficiently nice stable monoidal model category, and \mathcal{O} is a \mathfrak{C} -colored operad in \mathcal{M} such that cofibrant Smith \mathcal{O} -ideals are also entrywise cofibrant in the arrow category of \mathcal{M} with the projective model structure. Then there is a Quillen equivalence*

$$\{\text{Smith } \mathcal{O}\text{-Ideals}\} \xrightleftharpoons[\text{ker}]{\text{coker}} \{\mathcal{O}\text{-Algebra Maps}\}$$

induced by the cokernel and the kernel.

For example, this Theorem holds in the following situations:

- (1) \mathcal{O} is an arbitrary \mathfrak{C} -colored operad, and \mathcal{M} is either (i) the category of bounded or unbounded chain complexes over a field of characteristic zero (Corollary 5.2.4) or (ii) the stable module category of $k[G]$ -modules for some field k whose characteristic divides the order of G (Corollary 6.2.4).
- (2) \mathcal{O} is the commutative operad, and \mathcal{M} is the category of symmetric spectra with the positive flat stable model structure (Example 5.1.3).
- (3) \mathcal{O} is $\Sigma_{\mathfrak{C}}$ -cofibrant (e.g., the associative operad, A_∞ -operads, E_∞ -operads, and E_n -operads), and \mathcal{M} is the category of symmetric spectra with either the positive stable model structure or the positive flat stable model structure (Corollary 5.2.3 and Example 5.2.5).

The rest of this paper is organized as follows. In Section 2 we recall some basic facts about model categories and arrow categories. In Section 3 we define Smith ideals for an operad and prove that, when \mathcal{M} is pointed, there is an adjunction between Smith \mathcal{O} -ideals and \mathcal{O} -algebra maps given by the cokernel and the kernel. In Section 4 we define the model structures on Smith \mathcal{O} -ideals and \mathcal{O} -algebra maps and prove the Theorem above. In Section 5 we apply the Theorem to the commutative operad and $\Sigma_{\mathfrak{C}}$ -cofibrant operads. In Section 6 we apply the Theorem to entrywise cofibrant operads.

Acknowledgments. The authors would like to thank Adeel Khan, Tyler Lawson, and Denis Nardin for an email exchange about this project.

2. MODEL STRUCTURES ON THE ARROW CATEGORY

In this section we recall a few facts about monoidal model categories and arrow categories. Our main references for model categories are [Hir03, Hov99, SS00]. In this paper, $(M, \otimes, \mathbb{1}, \text{Hom})$ will usually be a bicomplete symmetric monoidal closed category [Mac98] (VII.7) with monoidal unit $\mathbb{1}$, internal hom Hom , initial object \emptyset , and terminal object $*$.

2.1. Monoidal Model Categories. A model category is *cofibrantly generated* if there are a set I of cofibrations and a set J of trivial cofibrations (i.e. maps which are both cofibrations and weak equivalences) which permit the small object argument (with respect to some cardinal κ), and a map is a (trivial) fibration if and only if it satisfies the right lifting property with respect to all maps in J (resp. I).

Let $I\text{-cell}$ denote the class of transfinite compositions of pushouts of maps in I , and let $I\text{-cof}$ denote retracts of such. In order to run the small object argument, we will assume the domains K of the maps in I (and J) are κ -small relative to $I\text{-cell}$ (resp. $J\text{-cell}$); i.e., given a regular cardinal $\lambda \geq \kappa$ and any λ -sequence $X_0 \rightarrow X_1 \rightarrow \dots$ formed of maps $X_\beta \rightarrow X_{\beta+1}$ in $I\text{-cell}$, the map of sets

$$\text{colim}_{\beta < \lambda} M(K, X_\beta) \longrightarrow M(K, \text{colim}_{\beta < \lambda} X_\beta)$$

is a bijection. An object is *small* if there is some κ for which it is κ -small. A *strongly cofibrantly generated* model category is a cofibrantly generated model category in which the domains of I and J are small with respect to the entire category.

Definition 2.1.1. A symmetric monoidal closed category M equipped with a model structure is called a *monoidal model category* if it satisfies the following *pushout product axiom* [SS00] (3.1):

- Given any cofibrations $f : X_0 \rightarrow X_1$ and $g : Y_0 \rightarrow Y_1$, the pushout product map

$$(X_0 \otimes Y_1) \coprod_{X_0 \otimes Y_0} (X_1 \otimes Y_0) \xrightarrow{f \square g} X_1 \otimes Y_1$$

is a cofibration. If, in addition, either f or g is a weak equivalence then $f \square g$ is a trivial cofibration.

2.2. Quillen Adjunctions and Quillen Equivalences. An adjunction with left adjoint L and right adjoint R is denoted by $L \dashv R$.

Definition 2.2.1. A *lax monoidal functor* $F : M \rightarrow N$ between two monoidal categories is a functor equipped with structure maps

$$FX \otimes FY \xrightarrow{F_{X,Y}^2} F(X \otimes Y), \quad \mathbb{1}^N \xrightarrow{F^0} F\mathbb{1}^M$$

for X and Y in M that are associative and unital in a suitable sense [Mac98] (XI.2). If, furthermore, M and N are symmetric monoidal categories, and F^2 is compatible

with the symmetry isomorphisms, then F is called a *lax symmetric monoidal functor*. If the structure maps F^2 and F^0 are isomorphisms (resp., identity maps), then F is called a *strong monoidal functor* (resp., *strict monoidal functor*).

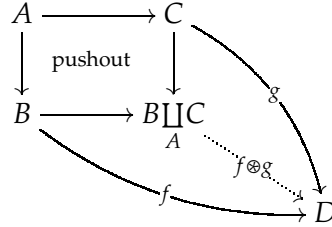
Definition 2.2.2. Suppose $L : \mathcal{M} \rightleftarrows \mathcal{N} : R$ is an adjunction between (semi) model categories.

- (1) We call $L \dashv R$ a *Quillen adjunction* if the right adjoint R preserves fibrations and trivial fibrations. In this case, we call L a *left Quillen functor* and R a *right Quillen functor*.
- (2) We call a Quillen adjunction $L \dashv R$ a *Quillen equivalence* if, for each map $f : LX \rightarrow Y \in \mathcal{N}$ with X cofibrant in \mathcal{M} and Y fibrant in \mathcal{N} , f is a weak equivalence in \mathcal{N} if and only if its adjoint $f^\# : X \rightarrow RY$ is a weak equivalence in \mathcal{M} .

2.3. Arrow Category. We now recall the two monoidal structures on the arrow category from [Hov ∞].

Definition 2.3.1. Suppose $(\mathcal{M}, \otimes, \mathbb{1})$ is a symmetric monoidal category with pushouts.

- (1) Given a solid-arrow commutative diagram



in \mathcal{M} in which the square is a pushout, the unique dotted induced map—i.e., the pushout corner map—will be denoted by $f \otimes g$. The only exception to this notation is when the pushout corner map is actually a pushout product of two maps, in which case we keep the box notation in Def. 2.1.1.

- (2) The *arrow category* $\vec{\mathcal{M}}$ is the category whose objects are maps in \mathcal{M} , in which a map $\alpha : f \rightarrow g$ is a commutative square

$$\begin{array}{ccc} X_0 & \xrightarrow{\alpha_0} & Y_0 \\ f \downarrow & & \downarrow g \\ X_1 & \xrightarrow{\alpha_1} & Y_1 \end{array} \quad (2.3.2)$$

in \mathcal{M} . We will also write $\text{Ev}_0 f = X_0$, $\text{Ev}_1 f = X_1$, $\text{Ev}_0 \alpha = \alpha_0$, and $\text{Ev}_1 \alpha = \alpha_1$.

- (3) The *tensor product monoidal structure* on $\vec{\mathcal{M}}$ is given by the monoidal product

$$X_0 \otimes X_1 \xrightarrow{f \otimes g} Y_0 \otimes Y_1$$

for $f : X_0 \rightarrow X_1$ and $g : Y_0 \rightarrow Y_1$. The arrow category equipped with this monoidal structure is denoted by $\vec{\mathcal{M}}^\otimes$. The monoidal unit is $\text{Id} : \mathbb{1} \rightarrow \mathbb{1}$.

- (4) The *pushout product monoidal structure* on \vec{M} is given by the pushout product

$$(X_0 \otimes Y_1) \coprod_{X_0 \otimes Y_0} (X_1 \otimes Y_0) \xrightarrow{f \square g} X_1 \otimes Y_1$$

for $f : X_0 \rightarrow X_1$ and $g : Y_0 \rightarrow Y_1$. The arrow category equipped with this monoidal structure is denoted by \vec{M}^\square . The monoidal unit is $\emptyset \rightarrow \mathbb{1}$.

- (5) Defining $L_0(X) = (\text{Id} : X \rightarrow X)$ and $L_1(X) = (\emptyset \rightarrow X)$ for $X \in M$, there are adjunctions

$$M \xrightleftharpoons[\text{Ev}_0]{L_0} \vec{M}^\otimes \qquad M \xrightleftharpoons[\text{Ev}_1]{L_1} \vec{M}^\square \quad (2.3.3)$$

with left adjoints on top and all functors strict symmetric monoidal.

2.4. Injective Model Structure. The following result about the injective model structure is from [Hov ∞] (2.1 and 2.2).

Theorem 2.4.1. *Suppose M is a model category.*

- (1) *There is a model structure on \vec{M} , called the injective model structure, in which a map $\alpha : f \rightarrow g$ as in (2.3.2) is a weak equivalence (resp., cofibration) if and only if α_0 and α_1 are weak equivalences (resp., cofibrations) in M . A map α is a (trivial) fibration if and only if α_1 and the pullback corner map*

$$X_0 \longrightarrow X_1 \times_{Y_1} Y_0$$

are (trivial) fibrations in M . Note that this implies that α_0 is also a (trivial) fibration. The arrow category equipped with the injective model structure is denoted by \vec{M}_{inj} .

- (2) *If M is cofibrantly generated, then so is \vec{M}_{inj} .*
 (3) *If M is a monoidal model category, then \vec{M}^\otimes equipped with the injective model structure is a monoidal model category, denoted $\vec{M}_{\text{inj}}^\otimes$.*

2.5. Projective Model Structure. The following result about the projective model structure is from [Hov ∞] (3.1).

Theorem 2.5.1. *Suppose M is a model category.*

- (1) *There is a model structure on \vec{M} , called the projective model structure, in which a map $\alpha : f \rightarrow g$ as in (2.3.2) is a weak equivalence (resp., fibration) if and only if α_0 and α_1 are weak equivalences (resp., fibrations) in M . A map α is a (trivial) cofibration if and only if α_0 and the pushout corner map*

$$X_1 \coprod_{X_0} Y_0 \xrightarrow{\alpha_1 \otimes g} Y_1$$

are (trivial) cofibrations in M . Note that this implies that α_1 is also a (trivial) cofibration. The arrow category equipped with the projective model structure is denoted by \vec{M}_{proj} .

- (2) If M is cofibrantly generated, then so is \vec{M}_{proj} .
- (3) If M is a monoidal model category, then \vec{M}^\square equipped with the projective model structure is a monoidal model category, denoted $\vec{M}_{\text{proj}}^\square$.

Remark 2.5.2. In the last statement in Theorem 2.5.1, Hovey [Hov ∞] (3.1) had the additional assumption that M be cofibrantly generated. However, the authors proved in [WY ∞ 3] that if M is a monoidal model category, then so is $\vec{M}_{\text{proj}}^\square$.

Proposition 2.5.3. Suppose M is a cofibrantly generated model category in which the domains and the codomains of all the generating cofibrations and the generating trivial cofibrations are small in M . Then \vec{M}_{inj} and \vec{M}_{proj} are both strongly cofibrantly generated model categories.

Proof. The generating (trivial) cofibrations in \vec{M}_{inj} are the maps $L_1 i$ and the maps

$$\begin{array}{ccc} A & \xrightarrow{i} & B \\ i \downarrow & & \downarrow = \\ B & \xrightarrow{=} & B \end{array}$$

for $i \in I$ (resp., $i \in J$) [Hov ∞] (2.2). The generating (trivial) cofibrations in \vec{M}_{proj} are the maps $L_0 I \cup L_1 I$ (resp., $L_0 J \cup L_1 J$). So the smallness of the domains of the generating (trivial) cofibrations in \vec{M}_{inj} and \vec{M}_{proj} follows from our assumption on the domains and the codomains in I and J . \square

3. SMITH IDEALS FOR OPERADS

Suppose $(M, \otimes, \mathbb{1})$ is a cocomplete symmetric monoidal category in which the monoidal product commutes with colimits on both sides, which is automatically true if M is a closed symmetric monoidal category. In this section we define Smith ideals for an arbitrary colored operad \mathcal{O} in M . When M is pointed, we observe in Theorem 3.4.2 that the cokernel and the kernel induce an adjunction between the categories of Smith \mathcal{O} -ideals and of \mathcal{O} -algebra maps. This will set the stage for the study of the homotopy theory of Smith \mathcal{O} -ideals in the next several sections.

3.1. Operads, Algebras, and Bimodules. The following material on profiles and colored symmetric sequences is from [YJ15]. For colored operads our references are [Yau16] and [WY ∞ 1].

Definition 3.1.1. Suppose \mathcal{C} is a set, whose elements will be called *colors*.

- (1) A \mathfrak{C} -profile is a finite, possibly empty sequence $\underline{c} = (c_1, \dots, c_n)$ with each $c_i \in \mathfrak{C}$.
- (2) When permutations act on \mathfrak{C} -profiles from the left (resp., right), the resulting groupoid is denoted by $\Sigma_{\mathfrak{C}}$ (resp., $\Sigma_{\mathfrak{C}}^{\text{op}}$).
- (3) The category of \mathfrak{C} -colored symmetric sequences in M is the diagram category $M^{\Sigma_{\mathfrak{C}}^{\text{op}} \times \mathfrak{C}}$. For a \mathfrak{C} -colored symmetric sequence X , we think of $\Sigma_{\mathfrak{C}}^{\text{op}}$ (resp., \mathfrak{C}) as parametrizing the inputs (resp., outputs). For $(\underline{c}; d) \in \Sigma_{\mathfrak{C}}^{\text{op}} \times \mathfrak{C}$, the corresponding entry of a \mathfrak{C} -colored symmetric sequence X is denoted by $X(\underline{c})^d$.
- (4) A \mathfrak{C} -colored operad $(\mathcal{O}, \gamma, 1)$ in M consists of:
 - a \mathfrak{C} -colored symmetric sequence \mathcal{O} in M ;
 - structure maps

$$\mathcal{O}(\underline{c})^d \otimes \bigotimes_{i=1}^n \mathcal{O}(\underline{b}_i)^{c_i} \xrightarrow{\gamma} \mathcal{O}(\underline{b})^d$$

in M for all $d \in \mathfrak{C}$, $\underline{c} = (c_1, \dots, c_n) \in \Sigma_{\mathfrak{C}}$, and $\underline{b}_i \in \Sigma_{\mathfrak{C}}$ for $1 \leq i \leq n$, where $\underline{b} = (\underline{b}_1, \dots, \underline{b}_n)$ is the concatenation of the \underline{b}_i 's;

- colored units $1_c : \mathbb{1} \longrightarrow \mathcal{O}(c)^c$ for $c \in \mathfrak{C}$.

This data is required to satisfy suitable associativity, unity, and equivariant conditions.

- (5) For a \mathfrak{C} -colored operad \mathcal{O} in M , an \mathcal{O} -algebra (A, λ) consists of:
 - objects $A_c \in M$ for $c \in \mathfrak{C}$;
 - structure maps

$$\mathcal{O}(\underline{c})^d \otimes A_{c_1} \otimes \dots \otimes A_{c_n} \xrightarrow{\lambda} A_d$$

in M for all $d \in \mathfrak{C}$ and $\underline{c} = (c_1, \dots, c_n) \in \Sigma_{\mathfrak{C}}$.

This data is required to satisfy suitable associativity, unity, and equivariant conditions. Maps of \mathcal{O} -algebras are required to preserve the structure maps. The category of \mathcal{O} -algebras in M is denoted by $\text{Alg}(\mathcal{O}; M)$.

- (6) Suppose (A, λ) is an \mathcal{O} -algebra for some \mathfrak{C} -colored operad \mathcal{O} in M . An A -bimodule (X, θ) consists of:
 - objects $X_c \in M$ for $c \in \mathfrak{C}$;
 - structure maps

$$\mathcal{O}(\underline{c})^d \otimes A_{c_1} \otimes \dots \otimes A_{c_{i-1}} \otimes X_{c_i} \otimes A_{c_{i+1}} \otimes \dots \otimes A_{c_n} \xrightarrow{\theta} X_d$$

in M for all $1 \leq i \leq n$ with $n \geq 1$, $d \in \mathfrak{C}$, and $\underline{c} = (c_1, \dots, c_n) \in \Sigma_{\mathfrak{C}}$.

This data is required to satisfy suitable associativity, unity, and equivariant conditions similar to those of an \mathcal{O} -algebra but with one input entry A and the output entry replaced by X . A map of A -bimodules is required to preserve the structure maps. The category of A -bimodules is denoted by $\text{Bimod}(A)$.

- (7) For a \mathfrak{C} -colored operad \mathcal{O} in M , we write

$$\vec{\mathcal{O}}^{\otimes} = L_0 \mathcal{O} \quad \text{and} \quad \vec{\mathcal{O}}^{\square} = L_1 \mathcal{O}$$

for the \mathfrak{C} -colored operads in \vec{M}^\otimes and \vec{M}^\square , respectively, where $L_0 : M \longrightarrow \vec{M}^\otimes$ and $L_1 : M \longrightarrow \vec{M}^\square$ are the strict monoidal functors in (2.3.3).

3.2. Arrow Category of Operadic Algebras.

Proposition 3.2.1. *Suppose \mathcal{O} is a \mathfrak{C} -colored operad in M . Then $\text{Alg}(\vec{\mathcal{O}}^\otimes; \vec{M}^\otimes)$ is canonically isomorphic to the arrow category of $\text{Alg}(\mathcal{O}; M)$.*

Proof. An $\vec{\mathcal{O}}^\otimes$ -algebra $f = \{f_c : X_c \longrightarrow Y_c\}$ consists of maps $f_c \in M$ for $c \in \mathfrak{C}$ and structure maps

$$\vec{\mathcal{O}}^\otimes(\underline{d}) \otimes \bigotimes_{i=1}^n f_{c_i} \xrightarrow{\lambda} f_d$$

in \vec{M}^\otimes for all $d \in \mathfrak{C}$ and $\underline{c} = (c_1, \dots, c_n) \in \Sigma_{\mathfrak{C}}$. This structure map is equivalent to the commutative square

$$\begin{array}{ccc} \mathcal{O}(\underline{d}) \otimes \bigotimes_{i=1}^n X_{c_i} & \xrightarrow{\lambda_0} & X_d \\ (\text{Id}, \otimes f_{c_i}) \downarrow & & \downarrow f_d \\ \mathcal{O}(\underline{d}) \otimes \bigotimes_{i=1}^n Y_{c_i} & \xrightarrow{\lambda_1} & Y_d \end{array}$$

in M . The associativity, unity, and equivariance of λ translate into those of λ_0 and λ_1 , making (X, λ_0) and (Y, λ_1) into \mathcal{O} -algebras in M . The commutativity of the previous square means $f : (X, \lambda_0) \longrightarrow (Y, \lambda_1)$ is a map of \mathcal{O} -algebras. \square

Remark 3.2.2. For the associative operad As , whose algebras are monoids, the identification of $\vec{\text{As}}^\otimes$ -algebras (i.e., monoids in \vec{M}^\otimes) with monoid maps in M is [Hov ∞] (1.5).

3.3. Operadic Smith Ideals.

Definition 3.3.1. Suppose \mathcal{O} is a \mathfrak{C} -colored operad in M . The category of *Smith \mathcal{O} -ideals* in M is defined as the category $\text{Alg}(\vec{\mathcal{O}}^\square; \vec{M}^\square)$.

Proposition 3.3.2. *Suppose \mathcal{O} is a \mathfrak{C} -colored operad in M . A Smith \mathcal{O} -ideal in M consists of precisely*

- an \mathcal{O} -algebra (Y, λ_1) in M ,
- a Y -bimodule (X, λ_0) in M , and
- a Y -bimodule map $f : (X, \lambda_0) \longrightarrow (Y, \lambda_1)$

such that, whenever $1 \leq i < j \leq n$, the diagram

$$\begin{array}{ccc}
 \mathcal{O}_{\underline{\mathfrak{C}}}^{(d)} \otimes Y_{c_1} \cdots Y_{c_{i-1}} X_{c_i} Y_{c_{i+1}} \cdots Y_{c_{j-1}} X_{c_j} Y_{c_{j+1}} \cdots Y_{c_n} & \xrightarrow{(\text{Id}, f_{c_j}, \text{Id})} & \mathcal{O}_{\underline{\mathfrak{C}}}^{(d)} \otimes Y_{c_1} \cdots Y_{c_{i-1}} X_{c_i} Y_{c_{i+1}} \cdots Y_{c_n} \\
 \downarrow (\text{Id}, f_{c_i}, \text{Id}) & & \downarrow \lambda_0 \\
 \mathcal{O}_{\underline{\mathfrak{C}}}^{(d)} \otimes Y_{c_1} \cdots Y_{c_{j-1}} X_{c_j} Y_{c_{j+1}} \cdots Y_{c_n} & \xrightarrow{\lambda_0} & X_d
 \end{array} \tag{3.3.3}$$

in \mathbf{M} is commutative. In the previous diagram, we omitted some \otimes to simplify the typography.

Proof. An $\vec{\mathcal{O}}^\square$ -algebra (f, λ) in $\vec{\mathbf{M}}^\square$ consists of

- maps $f_c : X_c \rightarrow Y_c$ in \mathbf{M} for $c \in \mathfrak{C}$;
- structure maps

$$\vec{\mathcal{O}}^\square(\underline{\mathfrak{C}}) \square f_{c_1} \square \cdots \square f_{c_n} \xrightarrow{\lambda} f_d$$

in $\vec{\mathbf{M}}^\square$ for all $d \in \mathfrak{C}$ and $\underline{\mathfrak{C}} = (c_1, \dots, c_n) \in \Sigma_{\mathfrak{C}}$

that are associative, unital, and equivariant. Since $\vec{\mathcal{O}}(\underline{\mathfrak{C}})$ is the map $\emptyset \rightarrow \mathcal{O}_{\underline{\mathfrak{C}}}^{(d)}$, when $n = 0$, the structure map λ is simply the map $\lambda_1 : \mathcal{O}_{\emptyset}^{(d)} \rightarrow Y_d$ in \mathbf{M} for $d \in \mathfrak{C}$. For $n \geq 1$, the structure map λ is equivalent to the commutative diagram

$$\begin{array}{ccc}
 \mathcal{O}_{\underline{\mathfrak{C}}}^{(d)} \otimes \text{dom}(f_{c_1} \square \cdots \square f_{c_n}) & \xrightarrow{\lambda_0} & X_d \\
 \downarrow (\text{Id}, f, \text{Id}) & & \downarrow f_d \\
 \mathcal{O}_{\underline{\mathfrak{C}}}^{(d)} \otimes Y_{c_1} \otimes \cdots \otimes Y_{c_n} & \xrightarrow{\lambda_1} & Y_d
 \end{array} \tag{3.3.4}$$

in \mathbf{M} . The domain of the iterated pushout product $f_{c_1} \square \cdots \square f_{c_n}$ is the colimit

$$\text{dom}(f_{c_1} \square \cdots \square f_{c_n}) = \text{colim}_{(\epsilon_1, \dots, \epsilon_n)} f_{\epsilon_1} \otimes \cdots \otimes f_{\epsilon_n} \tag{3.3.5}$$

in which $(\epsilon_1, \dots, \epsilon_n) \in \{0, 1\}^n \setminus \{(1, \dots, 1)\}$ and $f_{\epsilon_i} = X_{c_i}$ (resp., Y_{c_i}) if $\epsilon_i = 0$ (resp., 1). The maps that define the colimit are given by the f_{c_i} 's.

The bottom horizontal map λ_1 in (3.3.4) together with the maps $\lambda_1 : \mathcal{O}_{\emptyset}^{(d)} \rightarrow Y_d$ for $d \in \mathfrak{C}$ give Y the structure of an \mathcal{O} -algebra. The top horizontal map λ_0 , when restricted to

$$\mathcal{O}_{\underline{\mathfrak{C}}}^{(d)} \otimes Y_{c_1} \otimes \cdots \otimes Y_{c_{i-1}} \otimes X_{c_i} \otimes Y_{c_{i+1}} \otimes \cdots \otimes Y_{c_n}$$

for $1 \leq i \leq n$, gives X the structure of a Y -bimodule. The commutativity of (3.3.4) implies $f : (X, \lambda_0) \rightarrow (Y, \lambda_1)$ is a Y -bimodule map. The top horizontal map λ_0 together with the description (3.3.5) of the domain of $f_{c_1} \square \cdots \square f_{c_n}$ yield the commutative diagram (3.3.3).

The argument above can be reversed. In particular, to see that the commutative diagram (3.3.3) yields the top horizontal map λ_0 in (3.3.4), observe that the full subcategory of the punctured n -cube $\{0, 1\}^n \setminus \{(1, \dots, 1)\}$ consisting of $(\epsilon_1, \dots, \epsilon_n)$ with at most two 0's is a final subcategory [Mac98] (IX.3). \square

Remark 3.3.6. The special case of Prop. 3.3.2 for $\mathcal{O} = \text{As}$ is [Hov99] (1.7).

Remark 3.3.7. In the context of Prop. 3.3.2, a map

$$\left((X, \lambda_0) \xrightarrow{f} (Y, \lambda_1) \right) \xrightarrow{h} \left((X', \lambda'_0) \xrightarrow{f'} (Y', \lambda'_1) \right)$$

of Smith \mathcal{O} -ideals consists of

- a map $h_1 : Y \rightarrow Y'$ of \mathcal{O} -algebras;
- a map $h_0 : X \rightarrow X'$ of Y -bimodules, where X' becomes a Y -bimodule via restriction of scalars along h_1

such that the square

$$\begin{array}{ccc} X_c & \xrightarrow{h_{0c}} & X'_c \\ f_c \downarrow & & \downarrow f'_c \\ Y_c & \xrightarrow{h_{1c}} & Y'_c \end{array}$$

is commutative for each $c \in \mathfrak{C}$.

The description of Smith \mathcal{O} -ideals in Prop. 3.3.2 and their maps in the previous remark imply the following result.

Corollary 3.3.8. *Suppose \mathcal{O} is a \mathfrak{C} -colored operad in \mathbf{M} . Then there exists a $(\mathfrak{C} \sqcup \mathfrak{C})$ -colored operad \mathcal{O}^s in \mathbf{M} such that there is a canonical isomorphism of categories*

$$\text{Alg}(\vec{\mathcal{O}}^\square; \vec{\mathbf{M}}^\square) \cong \text{Alg}(\mathcal{O}^s; \mathbf{M}).$$

3.4. Operadic Smith Ideals and Maps of Operadic Algebras. Recall that a *0-object* in a category is an object that is both an initial object and a terminal object. A *pointed* category is a category with a 0-object. In [Hov ∞] (1.4) Hovey proved that, if \mathbf{M} is also pointed, then there is an adjunction

$$\vec{\mathbf{M}}^\square \xrightleftharpoons[\text{ker}]{\text{coker}} \vec{\mathbf{M}}^\otimes \quad (3.4.1)$$

with cokernel as the left adjoint and kernel as the right adjoint. The cokernel is strong symmetric monoidal that preserves the monoidal unit, and the kernel is lax symmetric monoidal. If \mathbf{M} is a pointed model category, then $(\text{coker}, \text{ker})$ is a Quillen adjunction [Hov ∞] (4.1). If \mathbf{M} is a stable model category, then it is a Quillen equivalence [Hov ∞] (4.3).

Theorem 3.4.2. *Suppose M is a cocomplete symmetric monoidal pointed category in which the monoidal product commutes with colimits on both sides. Suppose \mathcal{O} is a \mathfrak{C} -colored operad in M . Then the adjunction (3.4.1) induces an adjunction*

$$\mathrm{Alg}(\vec{\mathcal{O}}^\square; \vec{M}^\square) \xrightleftharpoons[\ker]{\mathrm{coker}} \mathrm{Alg}(\vec{\mathcal{O}}^\otimes; \vec{M}^\otimes) \quad (3.4.3)$$

in which both the left and the right adjoints are defined entrywise.

Proof. There is a solid-arrow commutative diagram

$$\begin{array}{ccc} \mathrm{Alg}(\vec{\mathcal{O}}^\square; \vec{M}^\square) & \xrightleftharpoons[\ker]{\mathrm{coker}} & \mathrm{Alg}(\vec{\mathcal{O}}^\otimes; \vec{M}^\otimes) \\ U \downarrow & & \downarrow U \\ (\vec{M}^\square)^\mathfrak{C} & \xrightleftharpoons[\ker]{\mathrm{coker}} & (\vec{M}^\otimes)^\mathfrak{C} \end{array} \quad (3.4.4)$$

with both U forgetful functors and $U \ker = \ker U$. The top horizontal kernel functor is well defined because $\ker : \vec{M}^\otimes \rightarrow \vec{M}^\square$ is lax symmetric monoidal and $\ker \vec{\mathcal{O}}^\otimes = \vec{\mathcal{O}}^\square$. Since \vec{M}^\otimes is cocomplete, so is the algebra category $\mathrm{Alg}(\vec{\mathcal{O}}^\otimes; \vec{M}^\otimes)$ [WY ∞ 1] (4.2.1). Moreover, since both vertical functors U are monadic [WY ∞ 1] (4.1.11), the Adjoint Lifting Theorem [Bor94] (4.5.6) implies that the top horizontal functor \ker admits a left adjoint. Normally this induced left adjoint is *not* lifted entrywise from the bottom horizontal left adjoint. However, since the cokernel functor is a unit-preserving strong symmetric monoidal functor and since $\mathrm{coker} \vec{\mathcal{O}}^\square = \vec{\mathcal{O}}^\otimes$, the induced left adjoint between the algebra categories is given entrywise by coker . \square

4. HOMOTOPY THEORY OF SMITH IDEALS FOR OPERADS

In this section, we study the homotopy theory of Smith ideals for an operad \mathcal{O} . Under suitable conditions on the underlying monoidal model category M , in Def. 4.2.3 we define model structures on the categories of Smith \mathcal{O} -ideals and of \mathcal{O} -algebra maps. When M is pointed, the cokernel and the kernel yield a Quillen adjunction between these model categories. Furthermore, in Theorem 4.4.1 we show that if M is stable and if cofibrant Smith \mathcal{O} -ideals are entrywise cofibrant in $\vec{M}_{\mathrm{proj}}^\square$, then the cokernel and the kernel yield a Quillen equivalence between the categories of Smith \mathcal{O} -ideals and of \mathcal{O} -algebra maps.

4.1. Admissibility of Operads. The following result is [WY ∞ 1] (6.1.1 and 6.1.3).

Theorem 4.1.1. *Suppose M is a strongly cofibrantly generated monoidal model category satisfying the following condition.*

(♣) : For each $n \geq 1$ and for each object $X \in \mathbf{M}^{\Sigma_n^{\text{op}}}$, the function

$$X \otimes_{\Sigma_n} (-)^{\square^n} : \mathbf{M} \longrightarrow \mathbf{M}$$

takes trivial cofibrations into some subclass of weak equivalences that is closed under transfinite composition and pushout.

Then each \mathfrak{C} -colored operad \mathcal{O} in \mathbf{M} is admissible. In other words, $\text{Alg}(\mathcal{O}; \mathbf{M})$ admits a cofibrantly generated model structure in which a map is a weak equivalence (resp., fibration) if and only if it is so in \mathbf{M} entrywise.

Example 4.1.2. Strongly cofibrantly generated monoidal model categories that satisfy (♣) include:

- (1) Pointed or unpointed simplicial sets [Qui67] and all of their left Bousfield localizations [Hir03].
- (2) Bounded or unbounded chain complexes over a field of characteristic zero [Qui67].
- (3) Symmetric spectra with either the positive stable model structure [MMSS01] or the positive flat stable model structure [Shi04].
- (4) The category of small categories with the folk model structure [Rez∞].
- (5) Simplicial modules over a field of characteristic zero [Qui67].
- (6) The stable module category of $k[G]$ -modules [Hov99] (2.2), where k is a field whose characteristic divides the order of the group G .

The condition (♣) for (1)–(3) is proved in [WY∞1] (Section 8), and (4)–(5) can be proved using similar arguments. The condition (♣) for the stable module category is proved by the argument in [WY∞2] (12.2). In each of these examples, the domains and the codomains of the generating (trivial) cofibrations are small with respect to the entire category. So Prop. 2.5.3 applies to show that in each case the arrow category with either the injective or the projective module structure is strongly cofibrantly generated.

4.2. Admissibility of Operads in the Arrow Category. Recall the injective model structure on the arrow category from Theorem 2.4.1.

Theorem 4.2.1. *If \mathbf{M} is a monoidal model category satisfying (♣), then so is $\vec{\mathbf{M}}_{\text{inj}}^{\otimes}$. Therefore, if \mathbf{M} is also cofibrantly generated in which the domains and the codomains of all the generating (trivial) cofibrations are small in \mathbf{M} , then every \mathfrak{C} -colored operad on $\vec{\mathbf{M}}_{\text{inj}}^{\otimes}$ is admissible.*

Proof. Suppose \mathbf{M} satisfies (♣) with respect to a subclass \mathcal{C} of weak equivalences that is closed under transfinite composition and pushout. We write \mathcal{C}' for the subclass of weak equivalences β in $\vec{\mathbf{M}}_{\text{inj}}^{\otimes}$ such that $\beta_0, \beta_1 \in \mathcal{C}$. Then \mathcal{C}' is closed under transfinite composition and pushout.

Suppose $f_X : X_0 \longrightarrow X_1$ is an object in $(\vec{M}^\otimes)^{\Sigma_n^{\text{op}}}$ and $\alpha : f_V \longrightarrow f_W$,

$$\begin{array}{ccc} V_0 & \xrightarrow{\alpha_0} & W_0 \\ f_V \downarrow & & \downarrow f_W \\ V_1 & \xrightarrow{\alpha_1} & W, \end{array} \quad (4.2.2)$$

is a trivial cofibration in $\vec{M}_{\text{inj}}^\otimes$. We will show that $f_X \otimes_{\Sigma_n} \alpha^{\square n}$ belongs to \mathcal{C}' . The map $f_X \otimes_{\Sigma_n} \alpha^{\square n}$ in \vec{M}^\otimes is the commutative square

$$\begin{array}{ccc} X_0 \otimes_{\Sigma_n} \text{dom}(\alpha_0^{\square n}) & \xrightarrow{X_0 \otimes_{\Sigma_n} \alpha_0^{\square n}} & X_0 \otimes_{\Sigma_n} W_0^{\otimes n} \\ f_X \otimes_{\Sigma_n} f_* \downarrow & & \downarrow f_X \otimes_{\Sigma_n} f_W^{\otimes n} \\ X_1 \otimes_{\Sigma_n} \text{dom}(\alpha_1^{\square n}) & \xrightarrow{X_1 \otimes_{\Sigma_n} \alpha_1^{\square n}} & X_1 \otimes_{\Sigma_n} W_1^{\otimes n} \end{array}$$

in M , where f_* is induced by f_V and f_W . Since α_0 and α_1 are trivial cofibrations in M and since $X_0, X_1 \in M^{\Sigma_n^{\text{op}}}$, the condition (\spadesuit) in M implies that the two horizontal maps in the previous diagram are both in \mathcal{C} . This shows that $\vec{M}_{\text{inj}}^\otimes$ satisfies (\spadesuit) with respect to the subclass \mathcal{C}' of weak equivalences.

The second assertion is now a consequence of Theorem 2.4.1, Prop. 2.5.3, and Theorem 4.1.1. \square

Definition 4.2.3. Suppose M is a cofibrantly generated monoidal model category satisfying (\spadesuit) in which the domains and the codomains of the generating (trivial) cofibrations are small with respect to the entire category. Suppose \mathcal{O} is a \mathfrak{C} -colored operad in M .

- (1) Equip the category of Smith \mathcal{O} -ideals $\text{Alg}(\vec{\mathcal{O}}^\square; \vec{M}^\square)$ with the model structure given by Cor. 3.3.8 and Theorem 4.1.1. In other words, a map α of Smith \mathcal{O} -ideals is a weak equivalence (resp., fibration) if and only if α_0 and α_1 are color-wise weak equivalences (resp., fibrations) in M .
- (2) Equip the category $\text{Alg}(\vec{\mathcal{O}}^\otimes; \vec{M}^\otimes)$ with the model structure given by Theorem 4.2.1. In other words, a map α in $\text{Alg}(\vec{\mathcal{O}}^\otimes; \vec{M}^\otimes)$ is a weak equivalence (resp., fibration) if and only if α^c (= the c -colored entry of α) is a weak equivalence (resp., fibration) in $\vec{M}_{\text{inj}}^\otimes$ for each $c \in \mathfrak{C}$.

Remark 4.2.4. In Def. 4.2.3 the model structure on Smith \mathcal{O} -ideals is induced by the forgetful functor to $M^{\mathfrak{C} \sqcup \mathfrak{C}}$, so its weak equivalences and fibrations are defined entrywise in M , or equivalently in $\vec{M}_{\text{proj}}^\square$. On the other hand, the model structure on \mathcal{O} -algebra maps $\text{Alg}(\vec{\mathcal{O}}^\otimes; \vec{M}^\otimes)$ is induced by the forgetful functor to $(\vec{M}_{\text{inj}}^\otimes)^\mathfrak{C}$. The (trivial) fibrations in $\text{Alg}(\vec{\mathcal{O}}^\otimes; \vec{M}^\otimes)$ are, in particular, entrywise (trivial) fibrations in

M. However, they are *not* defined entrywise in M, since (trivial) fibrations in $\vec{M}_{\text{inj}}^{\otimes}$ are not defined entrywise in M.

The following observation will be used in the proof of Theorem 6.2.1 below.

Proposition 4.2.5. *In the context of Def. 4.2.3, the model structure on Smith \mathcal{O} -ideals is cofibrantly generated with generating cofibrations $\vec{\mathcal{O}}^{\square} \circ (L_0 I \cup L_1 I)_c$ and generating trivial cofibrations $\vec{\mathcal{O}}^{\square} \circ (L_0 J \cup L_1 J)_c$ for $c \in \mathfrak{C}$, where I and J are the sets of generating cofibrations and generating trivial cofibrations in M. Here $(L_0 I \cup L_1 I)_c$ means the maps in $L_0 I \cup L_1 I \subseteq \vec{M}_{\text{proj}}^{\square}$ are regarded as maps in $(\vec{M}_{\text{proj}}^{\square})^{\mathfrak{C}}$ concentrated in a single color c with all other entries the initial object, and \circ is the circle product of the operad $\vec{\mathcal{O}}^{\square}$ [WY ∞ 1] (Section 3).*

Proof. The category $\text{Alg}(\vec{\mathcal{O}}^{\square}; \vec{M}^{\square})$ already has a model structure (namely, the one in Def. 4.2.3(1)) with weak equivalences and fibrations defined via the forgetful functor U in the free-forgetful adjunction

$$(\vec{M}_{\text{proj}}^{\square})^{\mathfrak{C}} \xrightleftharpoons[U]{\vec{\mathcal{O}}^{\square} \circ -} \text{Alg}(\vec{\mathcal{O}}^{\square}; \vec{M}^{\square}),$$

since the weak equivalences and fibrations in \vec{M}_{proj} are defined in M. Lemma 3.3 in [Y09] now implies that $\text{Alg}(\vec{\mathcal{O}}^{\square}; \vec{M}^{\square})$ has a cofibrantly generated model structure with weak equivalences and fibrations defined entrywise in \vec{M}_{proj} and with generating (trivial) cofibrations as stated above. Since a model structure is uniquely determined by the classes of weak equivalences and fibrations, this second model structure on $\text{Alg}(\vec{\mathcal{O}}^{\square}; \vec{M}^{\square})$ must be the same as the one in Def. 4.2.3(1). \square

4.3. Quillen Adjunction Between Operadic Smith Ideals and Algebra Maps.

Proposition 4.3.1. *Suppose M is a pointed cofibrantly generated monoidal model category satisfying (\spadesuit) in which the domains and the codomains of the generating (trivial) cofibrations are small with respect to the entire category. Suppose \mathcal{O} is a \mathfrak{C} -colored operad in M. Then the adjunction*

$$\text{Alg}(\vec{\mathcal{O}}^{\square}; \vec{M}^{\square}) \xrightleftharpoons[\text{ker}]{\text{coker}} \text{Alg}(\vec{\mathcal{O}}^{\otimes}; \vec{M}^{\otimes}) \quad (4.3.2)$$

in (3.4.3) is a Quillen adjunction.

Proof. Suppose α is a (trivial) fibration in $\text{Alg}(\vec{\mathcal{O}}^{\otimes}; \vec{M}^{\otimes})$. We must show that $\ker \alpha$ is a (trivial) fibration in $\text{Alg}(\vec{\mathcal{O}}^{\square}; \vec{M}^{\square})$, i.e., an entrywise (trivial) fibration in M. Since (trivial) fibrations in $\vec{M}_{\text{proj}}^{\square}$ are defined entrywise in M, it suffices to show that $U \ker \alpha$ is a (trivial) fibration in $(\vec{M}_{\text{proj}}^{\square})^{\mathfrak{C}}$. Since there is an equality (3.4.4)

$$U \ker \alpha = \ker U \alpha$$

and since $\ker : (\vec{M}_{\text{inj}}^{\otimes})^{\mathfrak{C}} \longrightarrow (\vec{M}_{\text{proj}}^{\square})^{\mathfrak{C}}$ is a right Quillen functor [Hov ∞] (4.1), we finish the proof by observing that $U\alpha \in (\vec{M}_{\text{inj}}^{\otimes})^{\mathfrak{C}}$ is a (trivial) fibration. \square

Recall that a pointed model category is *stable* if its homotopy category is a triangulated category [Hov99] (7.1.1).

Proposition 4.3.3. *In the setting of Prop. 4.3.1, suppose M is also a stable model category. Then the right Quillen functor \ker in (4.3.2) reflects weak equivalences between fibrant objects.*

Proof. Suppose α is a map in $\text{Alg}(\vec{\mathcal{O}}^{\otimes}; \vec{M}^{\otimes})$ between fibrant objects such that $\ker \alpha \in \text{Alg}(\vec{\mathcal{O}}^{\square}; \vec{M}^{\square})$ is a weak equivalence. So $\ker \alpha$ is entrywise a weak equivalence in M , or equivalently $U \ker \alpha \in (\vec{M}_{\text{proj}}^{\square})^{\mathfrak{C}}$ is a weak equivalence. We must show that α is a weak equivalence, i.e., that $U\alpha \in (\vec{M}_{\text{inj}}^{\otimes})^{\mathfrak{C}}$ is a weak equivalence. The map $U\alpha$ is still a map between fibrant objects, and

$$\ker U\alpha = U \ker \alpha$$

is a weak equivalence in $(\vec{M}_{\text{proj}}^{\square})^{\mathfrak{C}}$. Since $\ker : (\vec{M}_{\text{inj}}^{\otimes})^{\mathfrak{C}} \longrightarrow (\vec{M}_{\text{proj}}^{\square})^{\mathfrak{C}}$ is a right Quillen equivalence [Hov ∞] (4.3), it reflects weak equivalences between fibrant objects by [Hov99] (1.3.16). So $U\alpha$ is a weak equivalence. \square

4.4. Quillen Equivalence Between Operadic Smith Ideals and Algebra Maps. The following result says that, under suitable conditions, Smith \mathcal{O} -ideals and \mathcal{O} -algebra maps have equivalent homotopy theories.

Theorem 4.4.1. *Suppose M is a cofibrantly generated stable monoidal model category satisfying (\spadesuit) in which the domains and the codomains of the generating (trivial) cofibrations are small with respect to the entire category. Suppose \mathcal{O} is a \mathfrak{C} -colored operad in M such that cofibrant $\vec{\mathcal{O}}^{\square}$ -algebras are also underlying cofibrant in $(\vec{M}_{\text{proj}}^{\square})^{\mathfrak{C}}$. Then the Quillen adjunction*

$$\text{Alg}(\vec{\mathcal{O}}^{\square}; \vec{M}^{\square}) \xrightleftharpoons[\ker]{\text{coker}} \text{Alg}(\vec{\mathcal{O}}^{\otimes}; \vec{M}^{\otimes})$$

is a Quillen equivalence.

Proof. Using Prop. 4.3.3 and [Hov99] (1.3.16), it remains to show that for each cofibrant object $f_X \in \text{Alg}(\vec{\mathcal{O}}^{\square}; \vec{M}^{\square})$, the derived unit

$$f_X \xrightarrow{\eta} \ker R_{\mathcal{O}} \text{coker } f_X$$

is a weak equivalence in $\text{Alg}(\vec{\mathcal{O}}^{\square}; \vec{M}^{\square})$, where $R_{\mathcal{O}}$ is a fibrant replacement functor in $\text{Alg}(\vec{\mathcal{O}}^{\otimes}; \vec{M}^{\otimes})$. In other words, we must show that $U\eta$ is a weak equivalence in $(\vec{M}_{\text{proj}}^{\square})^{\mathfrak{C}}$.

Suppose R is a fibrant replacement functor in $(\vec{M}_{\text{inj}}^{\otimes})^{\mathcal{C}}$. Consider the solid-arrow commutative diagram

$$\begin{array}{ccc} U \operatorname{coker} f_X & \xrightarrow{\sim} & UR_{\mathcal{O}} \operatorname{coker} f_X \\ \downarrow \sim & \nearrow \alpha & \downarrow \\ RU \operatorname{coker} f_X & \xrightarrow{\twoheadrightarrow} & 0 \end{array}$$

in $(\vec{M}_{\text{inj}}^{\otimes})^{\mathcal{C}}$. Here the left vertical map is a trivial cofibration and is a fibrant replacement of $U \operatorname{coker} f_X$. The top horizontal map is a weak equivalence and is U applied to a fibrant replacement of $\operatorname{coker} f_X$. The other two maps are fibrations. So there is a dotted map α that makes the whole diagram commutative. By the 2-out-of-3 property, α is a weak equivalence between fibrant objects in $(\vec{M}_{\text{inj}}^{\otimes})^{\mathcal{C}}$. Since $\ker : (\vec{M}_{\text{inj}}^{\otimes})^{\mathcal{C}} \rightarrow (\vec{M}_{\text{proj}}^{\square})^{\mathcal{C}}$ is a right Quillen functor, by Ken Brown's Lemma [Hov99] (1.1.12) $\ker \alpha$ is a weak equivalence in $(\vec{M}_{\text{proj}}^{\square})^{\mathcal{C}}$.

We now have a commutative diagram

$$\begin{array}{ccccc} Uf_X & \xrightarrow{U\eta} & & & U \ker R_{\mathcal{O}} \operatorname{coker} f_X \\ \downarrow \varepsilon & & & & \uparrow = \\ \ker R \operatorname{coker} Uf_X & \xrightarrow{=} & \ker RU \operatorname{coker} f_X & \xrightarrow[\sim]{\ker \alpha} & \ker UR_{\mathcal{O}} \operatorname{coker} f_X \end{array}$$

in $(\vec{M}_{\text{proj}}^{\square})^{\mathcal{C}}$, where ε is the derived unit of Uf_X . To show that $U\eta$ is a weak equivalence, it suffices to show that ε is a weak equivalence. By assumption Uf_X is a cofibrant object in $(\vec{M}_{\text{proj}}^{\square})^{\mathcal{C}}$. Since $(\operatorname{coker}, \ker)$ is a Quillen equivalence between $(\vec{M}_{\text{proj}}^{\square})^{\mathcal{C}}$ and $(\vec{M}_{\text{inj}}^{\otimes})^{\mathcal{C}}$, the derived unit ε is a weak equivalence by [Hov99] (1.3.16). \square

Example 4.4.2. Among the model categories in Example 4.1.2,

- (1) the categories of bounded or unbounded chain complexes over a field of characteristic zero,
- (2) the category of symmetric spectra with either the positive stable model structure or the positive flat stable model structure, and
- (3) the stable module category of $k[G]$ -modules with the characteristic of k dividing the order of G

satisfy the hypotheses for M in Theorem 4.4.1. In what follows, when we mention symmetric spectra, we always assume that it is equipped with either the positive stable model structure or the positive flat stable model structure.

The condition about cofibrant Smith \mathcal{O} -ideals being color-wise cofibrant in $\vec{M}_{\text{proj}}^{\square}$ is more subtle. We will consider this issue in the next two sections.

5. SMITH IDEALS FOR COMMUTATIVE AND SIGMA-COFIBRANT OPERADS

In this section we apply Theorem 4.4.1 and consider Smith ideals for the commutative operad and $\Sigma_{\mathcal{C}}$ -cofibrant operads. In particular, in Corollary 5.2.3 we will show that Theorem 4.4.1 is applicable to all $\Sigma_{\mathcal{C}}$ -cofibrant operads. On the other hand, the commutative operad is usually not Σ -cofibrant. However, as we will see in Example 5.1.3, Theorem 4.4.1 is applicable to the commutative operad in symmetric spectra with the positive flat stable model structure.

5.1. Commutative Smith Ideals. For the commutative operad, which is entrywise the monoidal unit and whose algebras are commutative monoids, we use the following definition from [Whi ∞] (3.4).

Definition 5.1.1. A monoidal model category M is said to satisfy the *strong commutative monoid axiom* if $(-)^{\square^n}/\Sigma_n : M \longrightarrow M$ preserves cofibrations and trivial cofibrations.

The following result says that, under suitable conditions, commutative Smith ideals and commutative monoid maps have equivalent homotopy theories.

Corollary 5.1.2. *Suppose M is as in Theorem 4.4.1 that satisfies the strong commutative monoid axiom in which the monoidal unit is cofibrant. Then there is a Quillen equivalence*

$$\mathrm{Alg}(\overrightarrow{\mathrm{Com}}^{\square}; \overrightarrow{M}^{\square}) \xrightleftharpoons[\mathrm{ker}]{\mathrm{coker}} \mathrm{Alg}(\overrightarrow{\mathrm{Com}}^{\otimes}; \overrightarrow{M}^{\otimes})$$

in which Com is the commutative operad in M .

Proof. For the commutative operad, it is proved in [Whi ∞] (3.6 and 5.14) that, with the strong commutative monoid axiom and a cofibrant monoidal unit, cofibrant $\overrightarrow{\mathrm{Com}}^{\square}$ -algebras are also underlying cofibrant in $\overrightarrow{M}_{\mathrm{proj}}^{\square}$. So Theorem 4.4.1 applies. \square

Example 5.1.3 (Commutative Smith Ideals in Symmetric Spectra). The category of symmetric spectra with the positive flat stable model structure satisfies the hypotheses in Theorem 4.4.1 and the strong commutative monoid axiom [Whi ∞] (5.7) and has a cofibrant monoidal unit. Therefore, Corollary 5.1.2 applies to the commutative operad Com in symmetric spectra with the positive flat stable model structure. The same also holds if symmetric spectra is replaced by the category of chain complexes over a field of characteristic zero [Whi ∞] (5.1).

5.2. Smith Ideals for Sigma-Cofibrant Operads. For a cofibrantly generated model category M and a small category \mathcal{D} , recall that the diagram category $M^{\mathcal{D}}$ inherits a *projective model structure* with weak equivalences and fibrations defined entrywise in M [Hir03] (11.6.1).

Definition 5.2.1. For a cofibrantly generated model category M , a \mathcal{C} -colored operad in M is said to be $\Sigma_{\mathcal{C}}$ -cofibrant if its underlying \mathcal{C} -colored symmetric sequence is cofibrant. If \mathcal{C} is the one-point set, then we say Σ -cofibrant instead of $\Sigma_{\{*\}}$ -cofibrant

Proposition 5.2.2. *Suppose \mathcal{M} is a cofibrantly generated model category, and \mathcal{D} is a small category. If $X \in \mathcal{M}^{\mathcal{D}}$ is cofibrant, then $L_1 X \in (\vec{\mathcal{M}}_{\text{proj}}^{\square})^{\mathcal{D}}$ and $L_0 X \in (\vec{\mathcal{M}}_{\text{inj}}^{\otimes})^{\mathcal{D}}$ are cofibrant. In particular, if \mathcal{O} is a $\Sigma_{\mathfrak{C}}$ -cofibrant \mathfrak{C} -colored operad in \mathcal{M} , then $\vec{\mathcal{O}}^{\square} = L_1 \mathcal{O}$ is a $\Sigma_{\mathfrak{C}}$ -cofibrant \mathfrak{C} -colored operad in $\vec{\mathcal{M}}_{\text{proj}}^{\square}$.*

Proof. The Quillen adjunction $L_1 : \mathcal{M} \rightleftarrows \vec{\mathcal{M}}_{\text{proj}}^{\square} : \text{Ev}_1$ lifts to a Quillen adjunction of \mathcal{D} -diagram categories

$$\mathcal{M}^{\mathcal{D}} \begin{array}{c} \xrightarrow{L_1} \\ \xleftarrow{\text{Ev}_1} \end{array} (\vec{\mathcal{M}}_{\text{proj}}^{\square})^{\mathcal{D}}$$

by [Hir03] (11.6.5(1)), and similarly for (L_0, Ev_0) . \square

The following result says that, under suitable conditions, for a $\Sigma_{\mathfrak{C}}$ -cofibrant \mathfrak{C} -colored operad \mathcal{O} , Smith \mathcal{O} -ideals and \mathcal{O} -algebra maps have equivalent homotopy theories.

Corollary 5.2.3. *Suppose \mathcal{M} is as in Theorem 4.4.1, and \mathcal{O} is a $\Sigma_{\mathfrak{C}}$ -cofibrant \mathfrak{C} -colored operad in \mathcal{M} . Then cofibrant $\vec{\mathcal{O}}^{\square}$ -algebras are also underlying cofibrant in $(\vec{\mathcal{M}}_{\text{proj}}^{\square})^{\mathfrak{C}}$, so there is a Quillen equivalence*

$$\text{Alg}(\vec{\mathcal{O}}^{\square}; \vec{\mathcal{M}}^{\square}) \begin{array}{c} \xleftarrow{\text{coker}} \\ \xrightarrow{\text{ker}} \end{array} \text{Alg}(\vec{\mathcal{O}}^{\otimes}; \vec{\mathcal{M}}^{\otimes}).$$

Proof. The arrow category $\vec{\mathcal{M}}_{\text{proj}}^{\square}$ is a cofibrantly generated monoidal model category (Theorem 2.5.1). By Prop. 5.2.2 the \mathfrak{C} -colored operad $\vec{\mathcal{O}}^{\square}$ in $\vec{\mathcal{M}}_{\text{proj}}^{\square}$ is $\Sigma_{\mathfrak{C}}$ -cofibrant. The special case of [WY ∞ 1] (6.3.1(2)) applied to $\vec{\mathcal{O}}^{\square}$ now says that every cofibrant $\vec{\mathcal{O}}^{\square}$ -algebra is underlying cofibrant in $(\vec{\mathcal{M}}_{\text{proj}}^{\square})^{\mathfrak{C}}$. So Theorem 4.4.1 applies. \square

Corollary 5.2.4. *Suppose \mathcal{M} is the category of bounded or unbounded chain complexes over a field of characteristic zero. Then every \mathfrak{C} -colored operad in \mathcal{M} is $\Sigma_{\mathfrak{C}}$ -cofibrant. In particular, Corollary 5.2.3 is applicable for all colored operads in \mathcal{M} .*

Proof. Over a field of characteristic zero, Maschke's Theorem guarantees that every symmetric sequence is cofibrant. \square

Example 5.2.5. Suppose \mathcal{M} is as in Theorem 4.4.1, such as the categories in Example 4.4.2. Here are some examples of Σ -cofibrant operads, for which Corollary 5.2.3 is applicable.

Smith Ideals: The associative operad As , which has $\text{As}(n) = \coprod_{\Sigma_n} \mathbb{1}$ as the n th entry and which has monoids as algebras, is Σ -cofibrant. In this case, Corollary 5.2.3 is Hovey's Corollary 4.4(1) in [Hov ∞].

Smith A_{∞} -Ideals: Any A_{∞} -operad, defined as a Σ -cofibrant resolution of As , is Σ -cofibrant. In this case, Corollary 5.2.3 says that Smith A_{∞} -ideals and A_{∞} -algebra maps have equivalent homotopy theories. For instance, one

can take the standard differential graded A_∞ -operad [Mar96] and, for symmetric spectra, the Stasheff associahedra operad [Sta63].

Smith E_∞ -Ideals: Any E_∞ -operad, defined as a Σ -cofibrant resolution of the commutative operad Com , is Σ -cofibrant. In this case, Corollary 5.2.3 says that Smith E_∞ -ideals and E_∞ -algebra maps have equivalent homotopy theories. For example, for symmetric spectra one can take the Barratt-Eccles E_∞ -operad $E\Sigma_*$ [BE74].

Smith E_n -Ideals: For each $n \geq 1$ the little n -cube operad \mathcal{C}_n [BV73, May72] is Σ -cofibrant and is an E_n -operad by definition [Fre ∞] (4.1.13). In this case, with \mathbf{M} being symmetric spectra with the positive (flat) stable model structure, Corollary 5.2.3 says that Smith \mathcal{C}_n -ideals and \mathcal{C}_n -algebra maps have equivalent homotopy theories. One may also use other Σ -cofibrant E_n -operads [Fie ∞], such as the Fulton-MacPherson operad ([GJ ∞] and [Fre ∞] (4.3)), which is actually a cofibrant E_n -operad.

6. SMITH IDEALS FOR ENTRYWISE COFIBRANT OPERADS

In this section we apply Theorem 4.4.1 to operads that are not necessarily $\Sigma_{\mathcal{C}}$ -cofibrant. To do that, we need to redistribute some of the cofibrancy assumptions—that cofibrant Smith \mathcal{O} -ideals are underlying cofibrant in the arrow category—from the operad to the underlying category. We will show in Theorem 6.2.1 that Theorem 4.4.1 is applicable to all entrywise cofibrant operads provided \mathbf{M} satisfies the cofibrancy condition (\heartsuit) below. This implies that over the stable module category Theorem 4.4.1 is always applicable.

6.1. Cofibrancy Assumptions.

Definition 6.1.1. Suppose \mathbf{M} is a cofibrantly generated monoidal model category. Define the following conditions in \mathbf{M} .

(\heartsuit) : For each $n \geq 1$ and each map $f \in \mathbf{M}^{\Sigma_n^{\text{op}}}$ that is an underlying cofibration between cofibrant objects in \mathbf{M} , the function

$$f \square_{\Sigma_n} (-) : \mathbf{M}^{\Sigma_n} \longrightarrow \mathbf{M}$$

takes each map in \mathbf{M}^{Σ_n} that is an underlying cofibration in \mathbf{M} to a cofibration in \mathbf{M} .

$(\clubsuit)_{\text{cof}}$: For each $n \geq 1$ and each object $X \in \mathbf{M}^{\Sigma_n^{\text{op}}}$ that is underlying cofibrant in \mathbf{M} , the function

$$X \otimes_{\Sigma_n} (-)^{\square^n} : \mathbf{M} \longrightarrow \mathbf{M}$$

preserves cofibrations.

Remark 6.1.2. The condition (\heartsuit) implies $(\clubsuit)_{\text{cof}}$, since $(\emptyset \longrightarrow X) \square (-) = X \otimes (-)$. The condition $(\clubsuit)_{\text{cof}}$ was introduced in [WY ∞ 1] (6.2.1), where the authors proved that $(\clubsuit)_{\text{cof}}$ and its trivial cofibration variant are closely related to admissibility of

operads that are not necessarily $\Sigma_{\mathcal{C}}$ -cofibrant and the underlying cofibrancy of cofibrant operadic algebras. It is, therefore, no surprise that we consider $(\clubsuit)_{\text{cof}}$ and its variant (\heartsuit) here in order to use Theorem 4.4.1 for operads that are not necessarily $\Sigma_{\mathcal{C}}$ -cofibrant.

Proposition 6.1.3. *The condition (\heartsuit) holds in the following categories:*

- (1) *Simplicial sets with either the Quillen model structure or the Joyal model structure [Lur09];*
- (2) *Bounded or unbounded chain complexes over a field of characteristic zero;*
- (3) *Small categories with the folk model structure;*
- (4) *The stable module category of $k[G]$ -modules with the characteristic of k dividing the order of G .*

Proof. For simplicial sets with either model structure, a cofibration is precisely an injection, and the pushout product of two injections is again an injection. Dividing an injection by a Σ_n -action is still an injection. The other cases are proved similarly. \square

Proposition 6.1.4. *If (\heartsuit) holds in \mathbf{M} , then it also holds in any left Bousfield localization of \mathbf{M} .*

Proof. The condition (\heartsuit) only refers to cofibrations, which remain the same in any left Bousfield localization. \square

The next observation is the key that connects the cofibrancy condition (\heartsuit) in \mathbf{M} to the arrow category.

Theorem 6.1.5. *Suppose \mathbf{M} is a cofibrantly generated monoidal model category satisfying (\heartsuit) . Then the arrow category $\vec{\mathbf{M}}_{\text{proj}}^{\square}$ satisfies $(\clubsuit)_{\text{cof}}$.*

Proof. Suppose $f_X : X_0 \rightarrow X_1$ is an object in $(\vec{\mathbf{M}}_{\text{proj}}^{\square})^{\Sigma_n^{\text{op}}}$ that is underlying cofibrant in $\vec{\mathbf{M}}_{\text{proj}}^{\square}$. This means that f_X is a map in $\mathbf{M}^{\Sigma_n^{\text{op}}}$ that is an underlying cofibration between cofibrant objects in \mathbf{M} . The condition $(\clubsuit)_{\text{cof}}$ for $\vec{\mathbf{M}}_{\text{proj}}^{\square}$ asks that the function

$$f_X \square_{\Sigma_n} (-)^{\square_2 n} : \vec{\mathbf{M}}_{\text{proj}}^{\square} \rightarrow \vec{\mathbf{M}}_{\text{proj}}^{\square}$$

preserve cofibrations, where \square and \square_2 are the pushout products in \mathbf{M} and $\vec{\mathbf{M}}^{\square}$, respectively. When $n = 1$ the condition $(\clubsuit)_{\text{cof}}$ for $\vec{\mathbf{M}}_{\text{proj}}^{\square}$ is a special case of the pushout product axiom in $\vec{\mathbf{M}}_{\text{proj}}^{\square}$, which is true by Theorem A in [WY ∞ 3].

Next suppose $n \geq 2$ and $\alpha : f_V \longrightarrow f_W$ is a map in \vec{M} as in (4.2.2). The iterated pushout product $\alpha^{\square_{2n}} \in (\vec{M}^{\square})^{\Sigma_n}$ is the commutative square

$$\begin{array}{ccc} Z & \xrightarrow{\zeta_1} & Y_1 \\ \zeta_0 \downarrow & & \downarrow f_W^{\square_n} \\ Y_0 & \xrightarrow{\alpha_1^{\square_n}} & W_1^{\otimes n} \end{array} \quad (6.1.6)$$

in M^{Σ_n} for some object Z . Applying $f_X \square_{\Sigma_n} (-)$, the map $f_X \square_{\Sigma_n} \alpha^{\square_{2n}}$ is the commutative square

$$\begin{array}{ccc} \left[(X_1 Z) \amalg_{X_0 Z} (X_0 Y_0) \right]_{\Sigma_n} & \xrightarrow{\varphi} & \left[(X_1 Y_1) \amalg_{X_0 Y_1} (X_0 W_1^n) \right]_{\Sigma_n} \\ f_X \square_{\Sigma_n} \zeta_0 \downarrow & & \downarrow f_X \square_{\Sigma_n} f_W^{\square_n} \\ (X_1 Y_0)_{\Sigma_n} & \xrightarrow{(X_1 \alpha_1^{\square_n})_{\Sigma_n}} & (X_1 W_1^n)_{\Sigma_n} \end{array} \quad (6.1.7)$$

in M , where we omitted the tensor symbol \otimes to simplify the typography. Suppose α is a cofibration in $\vec{M}_{\text{proj}}^{\square}$; i.e., the map $\alpha_0 : V_0 \longrightarrow W_0$ and the pushout corner map $\alpha_1 \otimes f_W : V_1 \amalg_{V_0} W_0 \longrightarrow W_1$ are cofibrations in M . We must show that $f_X \square_{\Sigma_n} \alpha^{\square_{2n}}$ is a cofibration in $\vec{M}_{\text{proj}}^{\square}$. In other words, we must show that in (6.1.7):

- (1) $\varphi = \text{Ev}_0(f_X \square_{\Sigma_n} \alpha^{\square_{2n}})$ is a cofibration in M .
- (2) The pushout corner map of $f_X \square_{\Sigma_n} \alpha^{\square_{2n}}$ is a cofibration in M .

We will prove (1) and (2) in Lemmas 6.1.8 and 6.1.11, respectively. \square

Lemma 6.1.8. *The map φ in (6.1.7) is a cofibration in M .*

Proof. Taking Σ_n -coinvariants and taking pushouts commute by the commutation of colimits. So φ is also the induced map from the pushout of the top row to the pushout of the bottom row in the commutative diagram

$$\begin{array}{ccccc} (X_1 Z)_{\Sigma_n} & \xleftarrow{(f_X Z)_{\Sigma_n}} & (X_0 Z)_{\Sigma_n} & \xrightarrow{(X_0 \zeta_0)_{\Sigma_n}} & (X_0 Y_0)_{\Sigma_n} \\ (X_1 \zeta_1)_{\Sigma_n} \downarrow & & (X_0 \zeta_1)_{\Sigma_n} \downarrow & & \downarrow (X_0 \alpha_1^{\square_n})_{\Sigma_n} \\ (X_1 Y_1)_{\Sigma_n} & \xleftarrow{(f_X Y_1)_{\Sigma_n}} & (X_0 Y_1)_{\Sigma_n} & \xrightarrow{(X_0 f_W^{\square_n})_{\Sigma_n}} & (X_0 W_1^n)_{\Sigma_n} \end{array} \quad (6.1.9)$$

in M . Here the left square is commutative by definition, and the right square is $X_0 \otimes_{\Sigma_n} (-)$ applied to $\alpha^{\square_{2n}}$ in (6.1.6).

We consider the Reedy category D with three objects $\{-1, 0, 1\}$, a map $0 \rightarrow -1$ that lowers the degree, a map $0 \rightarrow 1$ that raises the degree, and no other non-identity maps. Using the Quillen adjunction [Hov99] (proof of 5.2.6)

$$M^D \begin{array}{c} \xrightarrow{\text{colim}} \\ \xleftarrow{\text{constant}} \end{array} M$$

to show that φ is a cofibration in M , it is enough to show that (6.1.9) is a Reedy cofibration in M^D . So we must show that in (6.1.9):

- (1) The left and the middle vertical arrows are cofibrations in M .
- (2) The pushout corner map of the right square is a cofibration in M .

The objects X_0 and X_1 in $M^{\Sigma_n^{\text{op}}}$ are cofibrant in M . The map $\zeta_1 = \text{Ev}_0(\alpha^{\square_{2^n}}) \in M^{\Sigma_n}$ is an underlying cofibration in M . Indeed, since $\alpha \in \vec{M}_{\text{proj}}^{\square}$ is a cofibration, so is the iterated pushout product $\alpha^{\square_{2^n}}$ by the pushout product axiom [WY03]. In particular, $\text{Ev}_0(\alpha^{\square_{2^n}})$ is a cofibration in M . The condition (\heartsuit) in M (for the map $\emptyset \rightarrow X_i$) now implies the left and the middle vertical maps $X_i \otimes_{\Sigma_n} \zeta_1$ in (6.1.9) are cofibrations in M .

Finally, since $X_0 \in M^{\Sigma_n^{\text{op}}}$ is cofibrant in M and since the pushout corner map of $\alpha^{\square_{2^n}} \in (\vec{M}_{\text{proj}}^{\square})^{\Sigma_n}$ is a cofibration in M , the condition (\heartsuit) in M again implies the pushout corner map of the right square $X_0 \otimes_{\Sigma_n} \alpha^{\square_{2^n}}$ in (6.1.9) is a cofibration in M . \square

Remark 6.1.10. In the previous proof, Ev_0 and \square_2 do not commute, so ζ_1 is not an iterated pushout product.

Lemma 6.1.11. *The pushout corner map of $f_X \square_{\Sigma_n} \alpha^{\square_{2^n}}$ in (6.1.7) is a cofibration in M .*

Proof. The pushout corner map of $f_X \square_{\Sigma_n} \alpha^{\square_{2^n}}$ is the map $f_X \square_{\Sigma_n} (\alpha_1^{\square_n} \otimes f_W^{\square_n})$. This is the Σ_n -coinvariants of the pushout product in the diagram

$$\begin{array}{ccc}
 X_0(Y_0 \coprod_Z Y_1) & \xrightarrow{(\text{Id}, \alpha_1^{\square_n} \otimes f_W^{\square_n})} & X_0 W_1^n \\
 (f_X, \text{Id}) \downarrow & \text{pushout} & \downarrow \\
 X_1(Y_0 \coprod_Z Y_1) & \rightarrow & [X_1(Y_0 \coprod_Z Y_1)] \coprod_{[X_0(Y_0 \coprod_Z Y_1)]} (X_0 W_1^n) \\
 & & \searrow f_X \square (\alpha_1^{\square_n} \otimes f_W^{\square_n}) \\
 & & X_1 W_1^n
 \end{array}$$

$(\text{Id}, \alpha_1^{\square_n} \otimes f_W^{\square_n})$

in M^{Σ_n} with $\alpha_1^{\square_n} \otimes f_W^{\square_n}$ the pushout corner map of $\alpha^{\square_{2^n}} \in (\vec{M}_{\text{proj}}^{\square})^{\Sigma_n}$ in (6.1.6). Since $\alpha^{\square_{2^n}}$ is a cofibration in $\vec{M}_{\text{proj}}^{\square}$, its pushout corner map $\alpha_1^{\square_n} \otimes f_W^{\square_n}$ is a cofibration in M . So the condition (\heartsuit) in M implies that $f_X \square_{\Sigma_n} (\alpha_1^{\square_n} \otimes f_W^{\square_n})$ is a cofibration in M . \square

6.2. Underlying Cofibrancy of Cofibrant Smith Ideals for Entrywise Cofibrant Operads.

Theorem 6.2.1. *Suppose M is a cofibrantly generated monoidal model category satisfying (\spadesuit) and (\heartsuit) in which the domains and the codomains of the generating (trivial) cofibrations are small with respect to the entire category. Suppose \mathcal{O} is an entrywise cofibrant \mathfrak{C} -colored operad in M . Then cofibrant Smith \mathcal{O} -ideals are underlying cofibrant in $(\vec{M}_{\text{proj}}^\square)^\mathfrak{C}$. In particular, if M is also stable, then there is a Quillen equivalence*

$$\text{Alg}(\vec{\mathcal{O}}^\square; \vec{M}^\square) \xrightleftharpoons[\text{ker}]{\text{coker}} \text{Alg}(\vec{\mathcal{O}}^\otimes; \vec{M}^\otimes).$$

Proof. Using Theorem 4.4.1 it is enough to prove the assertion that cofibrant Smith \mathcal{O} -ideals are underlying cofibrant in $(\vec{M}_{\text{proj}}^\square)^\mathfrak{C}$. Writing $\varnothing^{\vec{\mathcal{O}}^\square}$ for the initial $\vec{\mathcal{O}}^\square$ -algebra, first we claim that $\varnothing^{\vec{\mathcal{O}}^\square}$ is underlying cofibrant in $(\vec{M}_{\text{proj}}^\square)^\mathfrak{C}$. Indeed, for each color $d \in \mathfrak{C}$, the d -colored entry of the initial $\vec{\mathcal{O}}^\square$ -algebra is the object

$$\varnothing_d^{\vec{\mathcal{O}}^\square} = \vec{\mathcal{O}}^\square(d) = (\varnothing^M \longrightarrow \mathcal{O}(d_\varnothing))$$

in $\vec{M}_{\text{proj}}^\square$, where \varnothing^M is the initial object in M and the symbol \varnothing in (d_\varnothing) is the empty \mathfrak{C} -profile. Since \mathcal{O} is assumed entrywise cofibrant, it follows that each entry of the initial $\vec{\mathcal{O}}^\square$ -algebra $\varnothing^{\vec{\mathcal{O}}^\square}$ is underlying cofibrant in $\vec{M}_{\text{proj}}^\square$.

By Prop. 4.2.5 the model structure on $\text{Alg}(\vec{\mathcal{O}}^\square; \vec{M}^\square)$ is right-induced by the forgetful functor U to $(\vec{M}_{\text{proj}}^\square)^\mathfrak{C}$ and is cofibrantly generated by $\vec{\mathcal{O}}^\square \circ (L_0 I \cup L_1 I)_c$ and $\vec{\mathcal{O}}^\square \circ (L_0 J \cup L_1 J)_c$ for $c \in \mathfrak{C}$, where I and J are the generating (trivial) cofibrations in M . Suppose A is a cofibrant $\vec{\mathcal{O}}^\square$ -algebra. We must show that A is underlying cofibrant in $(\vec{M}_{\text{proj}}^\square)^\mathfrak{C}$. By [Hir03] (11.2.2) the cofibrant $\vec{\mathcal{O}}^\square$ -algebra A is the retract of the colimit of a transfinite composition, starting with $\varnothing^{\vec{\mathcal{O}}^\square}$, of pushouts of maps in $\vec{\mathcal{O}}^\square \circ (L_0 I \cup L_1 I)_c$ for $c \in \mathfrak{C}$. Since $\varnothing^{\vec{\mathcal{O}}^\square}$ is underlying cofibrant in $\vec{M}_{\text{proj}}^\square$ and since the class of cofibrations in a model category, such as $(\vec{M}_{\text{proj}}^\square)^\mathfrak{C}$, is closed under transfinite compositions [Hir03] (10.3.4), the following Lemma will finish the proof. \square

Lemma 6.2.2. *Under the hypotheses of Theorem 6.2.1, suppose $\alpha : f \longrightarrow g$ is a map in $(L_0 I \cup L_1 I)_c$ for some color $c \in \mathfrak{C}$, and*

$$\begin{array}{ccc} \vec{\mathcal{O}}^\square \circ f & \longrightarrow & B_0 \\ \vec{\mathcal{O}}^\square \circ \alpha \downarrow & & \downarrow j \\ \vec{\mathcal{O}}^\square \circ g & \longrightarrow & B_\infty \end{array}$$

is a pushout in $\text{Alg}(\vec{\mathcal{O}}^\square; \vec{M}^\square)$ with B_0 cofibrant and $UB_0 \in (\vec{M}_{\text{proj}}^\square)^\mathfrak{C}$ cofibrant. Then Uj is a cofibration in $(\vec{M}_{\text{proj}}^\square)^\mathfrak{C}$. In particular, B_∞ is also cofibrant and $UB_\infty \in (\vec{M}_{\text{proj}}^\square)^\mathfrak{C}$ is cofibrant.

Proof. By the filtration in [WY ∞ 1] (4.3.16) and the fact that cofibrations are closed under pushouts, to show that $Uj \in (\vec{M}_{\text{proj}}^\square)^\mathfrak{C}$ is a cofibration, it is enough to show that for each $n \geq 1$ and each color $d \in \mathfrak{C}$ the map

$$\vec{\mathcal{O}}_{B_0}^\square \binom{d}{nc} \square \alpha^{\square_{2n}} \quad (6.2.3)$$

in $\vec{M}_{\text{proj}}^\square$ is a cofibration, where $nc = (c, \dots, c)$ is the \mathfrak{C} -profile with n copies of the color c . The object $\vec{\mathcal{O}}_{B_0}^\square$ is defined in [WY ∞ 1] (4.3.5), and $\alpha^{\square_{2n}}$ is the n -fold pushout product of α . Recall that $\vec{M}_{\text{proj}}^\square$ satisfies $(\clubsuit)_{\text{cof}}$ by Theorem 6.1.5 and that $\vec{\mathcal{O}}^\square$ is entrywise cofibrant in $\vec{M}_{\text{proj}}^\square$ because \mathcal{O} is entrywise cofibrant in M . The cofibrancy of $B_0 \in \text{Alg}(\vec{\mathcal{O}}^\square; \vec{M}^\square)$ and [WY ∞ 1] (6.2.4) applied to $\vec{\mathcal{O}}^\square$ now imply that $\vec{\mathcal{O}}_{B_0}^\square$ is entrywise cofibrant in $\vec{M}_{\text{proj}}^\square$. By the condition $(\clubsuit)_{\text{cof}}$ in $\vec{M}_{\text{proj}}^\square$ once again, we can conclude that the map (6.2.3) is a cofibration because α is a cofibration in $\vec{M}_{\text{proj}}^\square$. \square

Corollary 6.2.4. *Suppose M is the stable module category of $k[G]$ -modules for some field k whose characteristic divides the order of G . Then for each \mathfrak{C} -colored operad \mathcal{O} in M , there is a Quillen equivalence*

$$\text{Alg}(\vec{\mathcal{O}}^\square; \vec{M}^\square) \xrightleftharpoons[\text{ker}]{\text{coker}} \text{Alg}(\vec{\mathcal{O}}^\otimes; \vec{M}^\otimes).$$

Proof. The stable module category is a stable model category that satisfies the hypotheses of Theorem 6.2.1 in which every object is cofibrant. \square

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